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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

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LINCOLN LABORATORY

GENERATION OF THE LUNAR EPHEMERIS
ON AN ELECTRONIC COMPUTER

M. E. ASH

Group 63

LEXINGTON   MASSACHUSETTS

ABSTRACT

This report describes the method of integration that is used in a Lincoln Laboratory computer program (the Planetary Ephemeris Program) to determine as functions of time the position and velocity of the Moon and the partial derivatives of these quantities with respect to initial conditions. The method consists of numerically integrating the differential equations for the differences between the positions, velocities and partial derivatives in the true lunar orbit and in Brown's mean lunar orbit.

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GENERATION OF THE LUNAR EPHEMERIS
ON AN ELECTRONIC COMPUTER

I. INTRODUCTION

A computer program, called the Planetary Ephemeris Program (PEP), is being written at Lincoln Laboratory. The purpose of the program is to improve planetary and lunar ephemerides by using the results of radar and optical observations. In this report we describe the method of integration that is used in PEP to determine as functions of time the position and velocity of the Moon and the partial derivatives of these quantities with respect to initial conditions.

In Ref. 1 we described Encke's method of integration used in PEP for the planets and the Earth-Moon barycenter; we also presented Encke's method of integration for the Moon. However, this method is not satisfactory for the Moon, since the motion of the Moon deviates greatly from elliptic motion. In fact, the expression given in Brown's lunar theory for the mean lunar orbit[2] implies that the Moon approximately follows an ellipse whose ascending node moves backward along the ecliptic one revolution in 18.6 years and whose perigee advances one revolution in 6 years.

Our method of integration for the Moon utilizes Brown's mean lunar orbit rather than the initial osculating elliptic orbit of Encke's method of integration. Namely, let $(x^1, \ldots, x^6)$ denote the position and velocity in the true orbit of the Moon and let $(y^1, \ldots, y^6)$ denote the position and velocity in Brown's mean lunar orbit. We then numerically integrate the differential equations for the $\xi^k = x^k - y^k$, determining the x^k from the results of the integration by the fact that the y^k are known as functions of time. The partial derivatives $\partial x^k / \partial \beta^j$ of position and velocity with respect to initial osculating elliptic orbital elements $(\beta^1, \ldots, \beta^6)$ needed in fitting to observations are determined either (1) by assuming that these quantities are equal to the partial derivatives $\partial y^k / \partial \bar{\beta}^j$ of position and velocity in Brown's mean lunar orbit with respect to initial mean orbital elements $(\bar{\beta}^1, \ldots, \bar{\beta}^6)$ or (2) by numerically integrating the differential equations satisfied by the quantities $\eta_j^k = \partial x^k / \partial \beta^j - \partial y^k / \partial \bar{\beta}^j$. Option (1) might give results of the accuracy required by the least-squares process of fitting to observations. Not having to follow the exact procedure of (2) would save a great deal of computer time. The partial derivatives $\partial x^k / \partial \alpha$ of position and velocity with respect to parameters α that are not initial conditions (such as the mass of the Moon or the combined mass of the Earth and Moon) are determined by numerically integrating the differential equations for these quantities; since these equations are given in Ref. 3, we do not derive them in this report.

We intend to integrate the equations in a coordinate system referred to the mean equinox and equator of 1950.0. The use of this coordinate system dictates some of the manipulations we perform in Secs. III and IV on Brown's mean lunar orbit, since the reference angles for this orbit are given relative to the mean equinox and ecliptic of date.

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The differential equations of motion of a given body about a central body can be numerically integrated with arbitrarily given accuracy over an arbitrarily given period of time if a small enough interval of integration is used and if enough figures can be handled in the computations to prevent significant accumulation of round-off errors. By subtracting a mean orbit from the true orbit as we do in the case of the Moon, the requirements on the size of the interval of integration and on the number of figures needed are less stringent than if we just worked with the original equations of motion.

The electronic computer (an IBM 360 Model 67) we intend to use for the numerical integration has hardware floating point arithmetic operations with execution times of a few microseconds which handle 7, 16 or 32 decimal places. As presently envisioned, PEP will make certain crucial computations (such as taking a numerical integration step or calculating mean lunar orbit quantities) with 32 decimal place accuracy. Other computations (such as the determination of the positions of perturbing planets) will be done with 16 decimal place accuracy. If necessary, the design of PEP could be altered so that crucial computations could be carried out to 64 (or even more) decimal place accuracy. Such higher precision floating point operations would have to be programmed operations rather than machine operations and would thus require a great deal more computer time.

In any event, PEP can be designed to handle enough figures to prevent the significant accumulation of round-off errors. The only question is whether so small an interval of integration is needed that excessive computer time is used in integrating the motion of the Moon for centuries with the accuracy required by observations. This point can only be determined by computer experimentation. Because of the rapidity with which improvements are being made in electronic computers, such a calculation could very well be handled within a decade if not at the present time.

Since PEP is written in the Fortran IV language, only slight modification of the program will be necessary for its use in future computers. It is quite easy to insert in PEP the effect of additional forces, since it is the logic concerned with making a numerical integration step and manipulating input and output which makes the program intricate, not the specific terms on the right-hand sides of the equations.

II. EQUATIONS OF MOTION AND EQUATIONS FOR PARTIAL DERIVATIVES WITH RESPECT TO INITIAL CONDITIONS

We make the following definitions concerning subscripts:

s = Sun

e = Earth

m = Moon

c = Earth-Moon barycenter (center of mass of Earth-Moon system)

j = j^{th} planet ($j = 1, 2, 4, \ldots, 9$)   .

Let γ denote the gravitational constant and suppose that (x^1, x^2, x^3) is an inertial coordinate system. We make the following notational conventions:

$x_s^k = k^{th}$ coordinate of s, etc.

$x_{js}^k = x_j^k - x_s^k = k^{th}$ coordinate of j relative to s so that

$x_{js}^k = -x_{sj}^k$, etc.

$r_{sj} = r_{js}$ = distance between s and j, etc.

M_s = mass of s, etc.

$M_c = M_e + M_m$ = mass of Earth-Moon barycenter   .

Let F_e^k and F_m^k denote the components of force on the Earth and Moon, respectively, in addition to those due to the planetary and solar attractions. Then by Newton's laws of motion and gravity we have

$$\left.\begin{aligned}
\frac{d^2 x_e^k}{dt^2} &= \gamma M_m \frac{x_{me}^k}{r_{me}^3} + \gamma M_s \frac{x_{se}^k}{r_{es}^3} + \gamma \sum_j M_j \frac{x_{je}^k}{r_{je}^3} + \frac{1}{M_e} F_e^k \\
\frac{d^2 x_m^k}{dt^2} &= \gamma M_e \frac{x_{em}^k}{r_{me}^3} + \gamma M_s \frac{x_{sm}^k}{r_{ms}^3} + \gamma \sum_j M_j \frac{x_{jm}^k}{r_{jm}^3} + \frac{1}{M_m} F_m^k
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \quad (1)$$

Subtracting the first equation of (1) from the second equation of (1), we obtain

$$\frac{d^2 x_{me}^k}{dt^2} = -\gamma M_s \left(\frac{M_c}{M_s}\right) \frac{x_{me}^k}{r_{me}^3} + B^k + \Psi^k + \left(\frac{1}{M_m} F_m^k - \frac{1}{M_e} F_e^k\right) \quad , \quad k = 1, 2, 3 \quad (2)$$

where

$$\left.\begin{aligned}
B^k &= \gamma M_s \left(\frac{x_{es}^k}{r_{es}^3} - \frac{x_{ms}^k}{r_{ms}^3}\right) \\
\Psi^k &= \gamma M_s \sum_j \frac{M_j}{M_s} \left(\frac{x_{jm}^k}{r_{jm}^3} - \frac{x_{je}^k}{r_{je}^3}\right)
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \quad (3)$$

Let H^k denote that part of the $(F_m^k/M_m - F_e^k/M_e)$ term due to the higher harmonics in the gravitational potentials of the Earth and Moon, and hereafter let F_e^k and F_m^k denote the components of force due to effects other than these higher harmonics and the planetary and solar attractions. Then (2) can be put in the form

$$\left.\begin{aligned}
\frac{dx_{me}^k}{dt} &= x_{me}^{k+3} \\
\frac{dx_{me}^{k+3}}{dt} &= -\gamma M_s \left(\frac{M_c}{M_s}\right) \frac{x_{me}^k}{r_{me}^3} + B^k + \Psi^k + H^k + \left(\frac{1}{M_m} F_m^k - \frac{1}{M_e} F_e^k\right) \\
x_{me}^k &= x_{ome}^k \quad , \quad x_{me}^{k+3} = x_{ome}^{k+3} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \quad (4)$$

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which is exactly as written in Ref. 4. The expression given for H^k in Ref. 5 includes the effect of the second and third harmonics of the Earth and of the second harmonic of the Moon.

Let $(\beta_m^1, \ldots, \beta_m^6)$ denote the osculating elliptic orbital elements at time t_o of the orbit of the Moon about the Earth. Differentiating (4), we obtain[6]

$$\frac{d(\partial x_{me}^k / \partial \beta_m^j)}{dt} = \frac{\partial x_{me}^{k+3}}{\partial \beta_m^j}$$

$$\left. \begin{aligned} \frac{d(\partial x_{me}^{k+3} / \partial \beta_m^j)}{dt} &= \gamma M_s \left(\frac{M_c}{M_s}\right) \frac{1}{r_{me}^3} \left(\frac{3x_{me}^k}{r_{me}^2} \sum_{\ell=1}^{3} x_{me}^\ell \frac{\partial x_{me}^\ell}{\partial \beta_m^j} - \frac{\partial x_{me}^k}{\partial \beta_m^j}\right) \\ &\quad + \frac{\partial B^k}{\partial \beta_m^j} + \frac{\partial \Psi^k}{\partial \beta_m^j} + \frac{\partial H^k}{\partial \beta_m^j} + \frac{\partial}{\partial \beta_m^j} \left(\frac{1}{M_m} F_m^k - \frac{1}{M_c} F_e^k\right) \\ \frac{\partial x_{me}^k}{\partial \beta_m^j} &= \frac{\partial x_{ome}^k}{\partial \beta_m^j} \quad , \quad \frac{\partial x_{me}^{k+3}}{\partial \beta_m^j} = \frac{\partial \dot{x}_{ome}^{k+3}}{\partial \beta_m^j} \quad \text{when } t = t_o \end{aligned} \right\} \begin{aligned} k &= 1, 2, 3 \\ j &= 1, \ldots, 6 \end{aligned} \quad (5)$$

where the partial derivatives of B^k, Ψ^k, H^k are given in Ref. 7.

III. MOTION IN BROWN'S MEAN LUNAR ORBIT

In the following the unit of time t is measured in ephemeris days. Let

  i = inclination of Brown's mean lunar orbital plane on the mean ecliptic of date

  Ω = ascending node of Brown's mean lunar orbital plane on the mean ecliptic of date measured from the mean equinox of date along the ecliptic

  ω = argument of perigee of Brown's mean lunar orbit measured along the orbital plane from the ascending node on the mean ecliptic of date.

We then have[2]

$$\left. \begin{aligned} \sin\frac{i}{2} &= 0.044886967 \quad (i \approx 5°145) \\ \Omega &= 259°183275 - 0°0529539222 (t - t_*) \\ &\quad + 1°557 \times 10^{-12} (t - t_*)^2 + 5°0 \times 10^{-20} (t - t_*)^3 \\ \omega &= \Gamma' - \Omega = 75°146281 + 0°1643580025 (t - t_*) \\ &\quad - 9°296 \times 10^{-12} (t - t_*)^2 - 3°1 \times 10^{-19} (t - t_*)^3 \end{aligned} \right\} \quad (6)$$

where t_* is the time at the epoch 1900 January 0.5 = J.E.D. 2415020.0. If we measure angles in radians, then

$$\Omega = 4.52360151485 - 9.24220294234919 \times 10^{-4} (t - t_*)$$
$$+ 2.717477645355 \times 10^{-14} (t - t_*)^2 + 8.72664625997 \times 10^{-22} (t - t_*)^3$$
$$\omega = 1.31155002408 + 2.868588295626071 \times 10^{-3} (t - t_*)$$
$$- 1.6224580726539 \times 10^{-13} (t - t_*)^2$$
$$- 5.4105206811824 \times 10^{-21} (t - t_*)^3$$

$$\tag{7}$$

From these equations it easily follows that

$$\frac{d\Omega}{dt} = -9.24220294234919 \times 10^{-4} + 5.434955290710 \times 10^{-14} (t - t_*)$$
$$+ 2.617993877991 \times 10^{-21} (t - t_*)^2$$
$$\frac{d\omega}{dt} = 2.868588295626071 \times 10^{-3} - 3.2449161453178 \times 10^{-13} (t - t_*)$$
$$- 1.62315620435472 \times 10^{-20} (t - t_*)^2$$

$$\tag{8}$$

$$\frac{d^2\Omega}{dt^2} = 5.434955290710 \times 10^{-14} + 5.235987755982 \times 10^{-21} (t - t_*)$$
$$\frac{d^2\omega}{dt^2} = -3.2449161453178 \times 10^{-13} - 3.24631240870944 \times 10^{-20} (t - t_*)$$

$$\tag{9}$$

Let (v^1, v^2, v^3) be a coordinate system such that the v^1 axis points toward the perigee of the mean lunar orbit, the v^2 axis lies in the orbital plane and points in the direction of motion at perigee, and the v^3 axis is perpendicular to the orbital plane and completes the right-hand system. Let (w^1, w^2, w^3) be a coordinate system referred to the mean equinox and ecliptic of date, that is, a coordinate system such that the w^1 axis points toward the mean equinox of date, the w^3 axis is perpendicular to the mean ecliptic of date and points to the north, and the w^2 axis completes the right-hand system. The relation between the (v^1, v^2, v^3) and (w^1, w^2, w^3) coordinate systems, assuming that they have the same origin, is

$$w^j = \sum_{k=1}^{3} B_k^j v^k \quad , \quad v^j = \sum_{k=1}^{3} B_j^k w^k \quad , \quad j = 1, 2, 3 \tag{10}$$

where the orthogonal matrix $B = (B_k^j)$ is given by[8]

$$B_1^1 = \cos \Omega \, \cos \omega - \sin \Omega \, \sin \omega \, \cos i$$

$$B_2^1 = -\cos \Omega \, \sin \omega - \sin \Omega \, \cos \omega \, \cos i$$

$$B_3^1 = \sin \Omega \, \sin i$$

$$B_1^2 = \sin \Omega \, \cos \omega + \cos \Omega \, \sin \omega \, \cos i$$

$$B_2^2 = -\sin \Omega \, \sin \omega + \cos \Omega \, \cos \omega \, \cos i$$

$$B_3^2 = -\cos\Omega\,\sin i$$

$$B_1^3 = \sin\omega\,\sin i$$

$$B_2^3 = \cos\omega\,\sin i$$

$$B_3^3 = \cos i \quad . \tag{11}$$

We have

$$\left.
\begin{aligned}
\frac{dB_k^j}{dt} &= \frac{\partial B_k^j}{\partial\Omega}\frac{d\Omega}{dt} + \frac{\partial B_k^j}{\partial\omega}\frac{d\omega}{dt} \\[2ex]
\frac{d^2 B_k^j}{dt} &= \frac{\partial B_k^j}{\partial\Omega}\frac{d^2\Omega}{dt^2} + \frac{\partial B_k^j}{\partial\omega}\frac{d^2\omega}{dt^2} \\[2ex]
&\quad + \frac{\partial^2 B_k^j}{\partial\Omega^2}\left(\frac{d\Omega}{dt}\right)^2 + \frac{\partial^2 B_k^j}{\partial\omega^2}\left(\frac{d\omega}{dt}\right)^2 \; 2\,\frac{\partial^2 B_k^j}{\partial\Omega\,\partial\omega}\frac{d\Omega}{dt}\frac{d\omega}{dt}
\end{aligned}
\right\} \; j,k = 1,2,3 \tag{12}$$

where

$$\frac{\partial B_k^1}{\partial\Omega} = -B_k^2 \qquad \frac{\partial B_k^2}{\partial\Omega} = B_k^1 \qquad \frac{\partial B_k^3}{\partial\Omega} = 0 \quad , \quad k = 1,2,3 \tag{13}$$

$$\frac{\partial B_1^j}{\partial\omega} = B_2^j \qquad \frac{\partial B_2^j}{\partial\omega} = -B_1^j \qquad \frac{\partial B_3^j}{\partial\omega} = 0 \quad , \quad j = 1,2,3 \tag{14}$$

$$\frac{\partial^2 B_k^1}{\partial\Omega^2} = -B_k^1 \qquad \frac{\partial^2 B_k^2}{\partial\Omega^2} = -B_k^2 \qquad \frac{\partial^2 B_k^3}{\partial\Omega^2} = 0 \quad , \quad k = 1,2,3 \tag{15}$$

$$\frac{\partial^2 B_1^j}{\partial\omega^2} = -B_1^j \qquad \frac{\partial^2 B_2^j}{\partial\omega^2} = -B_2^j \qquad \frac{\partial^2 B_3^j}{\partial\omega^2} = 0 \quad , \quad j = 1,2,3 \tag{16}$$

$$\left.
\begin{aligned}
\frac{\partial^2 B_1^1}{\partial\Omega\,\partial\omega} &= -B_2^2 & \frac{\partial^2 B_1^2}{\partial\Omega\,\partial\omega} &= B_2^1 & \frac{\partial^2 B_k^3}{\partial\Omega\,\partial\omega} &= 0 \\[2ex]
\frac{\partial^2 B_2^1}{\partial\Omega\,\partial\omega} &= B_1^2 & \frac{\partial^2 B_2^2}{\partial\Omega\,\partial\omega} &= -B_1^1 & \frac{\partial^2 B_3^j}{\partial\Omega\,\partial\omega} &= 0
\end{aligned}
\right\} \; j,k = 1,2,3 \quad . \tag{17}$$

Let (y^1, y^2, y^3) be a coordinate system referred to the mean equinox and equator of 1950.0, that is, a coordinate system such that the y^1 axis points toward the mean equinox of 1950.0, the y^3 axis is perpendicular to the mean equator (of the Earth) of 1950.0, and the y^2 axis completes the right-hand system. The notation 1950.0 denotes the instant near the beginning of the calendar year 1950 when the longitude of the mean Sun was $18^h\,40^m$ so that 1950.0 is J. E. D. 2433282.423.[9] The relation between the (y^1, y^2, y^3) and (w^1, w^2, w^3) coordinate systems is given by

$$y^j = \sum_{k=1}^{3} A_k^j w^k \quad , \quad w^j = \sum_{k=1}^{3} A_j^k y^k \quad , \quad j = 1,2,3 \tag{18}$$

where expressions for the orthogonal matrix $A = (A_k^j)$ and its derivatives dA/dt and d^2A/dt^2 are given as functions of time in Appendix B.

Combining (10) and (18) we see that the relation between the (y^1, y^2, y^3) coordinate system referred to the mean equinox and equator of 1950.0 and the (v^1, v^2, v^3) coordinate system with v^1 axis pointed toward the perigee of Brown's mean lunar orbit and with v^3 axis perpendicular to Brown's mean lunar orbital plane pointed toward the north is

$$y^j = \sum_{k=1}^{3} C_k^j v^k \quad , \quad v^j = \sum_{k=1}^{3} C_j^k y^k \quad , \quad j = 1, 2, 3 \tag{19}$$

where the orthogonal matrix $C = AB$ is given by

$$C_k^j = \sum_{\ell=1}^{3} A_\ell^j B_k^\ell \quad , \quad j, k = 1, 2, 3 \quad . \tag{20}$$

The derivatives of the matrix C are

$$\left.\begin{aligned}
\frac{dC}{dt} &= \frac{dA}{dt} B + A \frac{dB}{dt} \\
\frac{d^2C}{dt^2} &= \frac{d^2A}{dt^2} B + 2 \frac{dA}{dt} \frac{dB}{dt} + A \frac{d^2B}{dt^2}
\end{aligned}\right\} \quad . \tag{21}$$

We are interested in a body moving in Brown's mean lunar orbit with position coordinates (v^1, v^2). The position, velocity and acceleration of this body in the coordinate system referred to the mean equinox and equator of 1950.0 is then

$$\left.\begin{aligned}
y^j &= \sum_{k=1}^{2} C_k^j v^k \\
\frac{dy^j}{dt} &= \sum_{k=1}^{2} C_k^j \frac{dv^k}{dt} + \sum_{k=1}^{2} \frac{dC_k^j}{dt} v^k \\
\frac{d^2y^j}{dt^2} &= \sum_{k=1}^{2} C_k^j \frac{d^2v^k}{dt^2} + 2 \sum_{k=1}^{2} \frac{dC_k^j}{dt} \frac{dv^k}{dt} + \sum_{k=1}^{2} \frac{d^2C_k^j}{dt^2} v^k
\end{aligned}\right\} \quad j = 1, 2, 3 \quad . \tag{22}$$

The mean anomaly L at time t in Brown's mean lunar orbit is[2]

$$L = \mathbb{C} - \Gamma' = -63°895392 + 13°0649924465 \, (t - t_*)$$
$$+ 6°889 \times 10^{-12} \, (t - t_*)^2 + 2°99 \times 10^{-19} \, (t - t_*)^3 \quad . \tag{23}$$

If we measure angles in radians, then

$$L = -1.1151849673 + 0.22802713493961401 \, (t - t_*)$$
$$+ 1.202357321699 \times 10^{-13} \, (t - t_*)^2 + 5.218534463463$$
$$\times 10^{-21} \, (t - t_*)^3 \quad . \tag{24}$$

From this it follows that

$$\frac{dL}{dt} = 0.22802713493961401 + 2.404714643398 \times 10^{-13} (t - t_*)$$
$$+ 1.5655603390389 \times 10^{-20} (t - t_*)^2$$
$$\frac{d^2L}{dt^2} = 2.404714643398 \times 10^{-13} + 3.1311206780778 \times 10^{-20} (t - t_*)$$

(25)

The eccentricity e of Brown's mean lunar orbit is constant:[2]

$$e = 0.054900489 \quad .$$

(26)

The semimajor axis a of Brown's mean lunar orbit is also constant. By inference from Ref. 2, we can assume that

$$a = 60.2665 \text{ equatorial earth radii} \quad .$$

(27)

Let u be the eccentric anomaly at time t in Brown's mean lunar orbit. It is determined by solving Kepler's equation

$$L = u - e \sin u$$

(28)

by iteration. Then the radius distance ρ and the components of position (v^1, v^2) in Brown's mean lunar orbit at time t are given by[10]

$$\rho = a(1 - e \cos u)$$
$$v^1 = a(\cos u - e)$$
$$v^2 = a \sqrt{1 - e^2} \sin u$$

(29)

From (28) it follows that

$$\frac{du}{dt} = \frac{1}{(1 - e \cos u)} \frac{dL}{dt}$$

(30)

so that by (29)

$$\frac{dv^1}{dt} = -\frac{a \sin u}{(1 - e \cos u)} \frac{dL}{dt}$$
$$\frac{dv^2}{dt} = \frac{a \sqrt{1 - e^2} \cos u}{(1 - e \cos u)} \frac{dL}{dt}$$

(31)

$$\frac{d^2v^1}{dt^2} = \frac{a}{(1 - e \cos u)} \left[-\sin u \frac{d^2L}{dt^2} - \frac{(\cos u - e)}{(1 - e \cos u)^2} \left(\frac{dL}{dt}\right)^2 \right]$$
$$\frac{d^2v^2}{dt^2} = \frac{a \sqrt{1 - e^2}}{(1 - e \cos u)} \left[\cos u \frac{d^2L}{dt^2} - \frac{\sin u}{(1 - e \cos u)^2} \left(\frac{dL}{dt}\right)^2 \right]$$

(32)

The position, velocity and acceleration in the coordinate system referred to the mean equinox and equator of 1950.0 is then given by (22).

IV. PARTIAL DERIVATIVES IN BROWN'S MEAN LUNAR ORBIT

Let

$\bar{a}$ = semimajor axis at time t_o of Brown's mean lunar orbit

$\bar{e}$ = eccentricity at time t_o of Brown's mean lunar orbit

$\bar{i}$ = inclination at time t_o of Brown's mean lunar orbit to the mean equator of 1950.0

$\bar{\Omega}$ = ascending node at time t_o of Brown's mean lunar orbit on the mean equator of 1950.0 measured from the mean equinox of 1950.0

$\bar{\omega}$ = argument of perigee at time t_o of Brown's mean lunar orbit measured from the ascending node on the mean equator of 1950.0

$\bar{l}$ = mean anomaly at the initial time t_o in Brown's mean lunar orbit.

In this section we shall derive the expressions for the partial derivatives of the position, velocity and acceleration in Brown's mean lunar orbit with respect to these quantities.

We of course have $\bar{a} = a$ and $\bar{e} = e$. Let

$$\mu = (\gamma M_s) \frac{M_c}{M_s} \quad . \tag{33}$$

Then the mean motion in Brown's mean lunar orbit is

$$n = \mu^{1/2} a^{-3/2} \quad . \tag{34}$$

If the orbital elements of Brown's mean lunar orbit were not functions of time, we would have the mean anomaly L at time t given by

$$L = \bar{l} + n(t - t_o) \quad .$$

By this equation and expression (24), the mean anomaly L at time t in Brown's mean lunar orbit can be written in the form

$$L = \bar{l} + n(t - t_o) + L_1(t - t_o) + L_2(t - t_o)^2 + L_3(t - t_o)^3 \quad .$$

so that

$$\left. \begin{array}{l} \dfrac{\partial L}{\partial \bar{a}} = -\dfrac{3}{2} \dfrac{n(t - t_o)}{a} \\[3mm] \dfrac{\partial L}{\partial \bar{e}} = 0 \\[3mm] \dfrac{\partial L}{\partial \bar{l}} = 1 \end{array} \right\} \quad . \tag{35}$$

Then from (28) it follows that

$$\left. \begin{array}{l} \dfrac{\partial u}{\partial \bar{a}} = -\dfrac{3}{2} \dfrac{n(t - t_o)}{a(1 - e \cos u)} \\[3mm] \dfrac{\partial u}{\partial \bar{e}} = \dfrac{\sin u}{(1 - e \cos u)} \\[3mm] \dfrac{\partial u}{\partial \bar{l}} = \dfrac{1}{(1 - e \cos u)} \end{array} \right\} \quad . \tag{36}$$

Differentiation of (29) with respect to $\bar{a}$, $\bar{e}$ and $\bar{l}$ gives

$$\frac{\partial v^1}{\partial \bar{a}} = (\cos u - e) + \frac{3}{2} \frac{n \sin u \,(t - t_o)}{(1 - e \cos u)}$$

$$\frac{\partial v^2}{\partial \bar{a}} = \sqrt{1 - e^2}\, \sin u - \frac{3}{2} \frac{n \sqrt{1 - e^2}\, \cos u \,(t - t_o)}{(1 - e \cos u)} \tag{37}$$

$$\frac{\partial v^1}{\partial \bar{e}} = -a - \frac{a \sin^2 u}{(1 - e \cos u)} \quad , \quad \frac{\partial v^2}{\partial \bar{e}} = -\frac{ae \sin u}{\sqrt{1 - e^2}} + \frac{a \sqrt{1 - e^2}\, \sin u \cos u}{(1 - e \cos u)} \tag{38}$$

$$\frac{\partial v^1}{\partial \bar{l}} = -\frac{a \sin u}{(1 - e \cos u)} \quad , \quad \frac{\partial v^2}{\partial \bar{l}} = \frac{a \sqrt{1 - e^2}\, \cos u}{(1 - e \cos u)} \quad . \tag{39}$$

For any parameter $\bar{\alpha}$ that is independent of time (such as the initial orbital elements), we have $\frac{d}{dt} \frac{\partial}{\partial \bar{\alpha}} = \frac{\partial}{\partial \bar{\alpha}} \frac{d}{dt}$. Thus, either differentiating Eqs. (31) and (32) with respect to $\bar{a}$, $\bar{e}$, $\bar{l}$ or differentiating Eqs. (37), (38) and (39) with respect to time, we obtain

$$\frac{d}{dt}\left(\frac{\partial v^1}{\partial \bar{a}}\right) = \frac{3}{2} \frac{n \sin u}{(1 - e \cos u)} - \frac{1}{(1 - e \cos u)} \frac{dL}{dt} \left[\sin u - \frac{3}{2} \frac{n(t - t_o)\,(\cos u - e)}{(1 - e \cos u)^2}\right]$$

$$\frac{d}{dt}\left(\frac{\partial v^2}{\partial \bar{a}}\right) = -\frac{3}{2} \frac{n \sqrt{1 - e^2}\, \cos u}{(1 - e \cos u)} + \frac{\sqrt{1 - e^2}}{(1 - e \cos u)} \frac{dL}{dt} \left[\cos u + \frac{3}{2} \frac{n(t - t_o)\,\sin u}{(1 - e \cos u)^2}\right] \tag{40}$$

$$\frac{d}{dt}\left(\frac{\partial v^1}{\partial \bar{e}}\right) = -\frac{a \sin u}{(1 - e \cos u)^2} \frac{dL}{dt} \left[\cos u + \frac{(\cos u - e)}{(1 - e \cos u)}\right]$$

$$\frac{d}{dt}\left(\frac{\partial v^2}{\partial \bar{e}}\right) = \frac{a}{(1 - e \cos u)} \frac{dL}{dt} \left\{-\frac{e \cos u}{\sqrt{1 - e^2}} + \frac{\sqrt{1 - e^2}}{(1 - e \cos u)} \left[\cos^2 u - \frac{\sin^2 u}{(1 - e \cos u)}\right]\right\} \tag{41}$$

$$\frac{d}{dt}\left(\frac{\partial v^1}{\partial \bar{l}}\right) = -\frac{a(\cos u - e)}{(1 - e \cos u)^3} \frac{dL}{dt} \quad , \quad \frac{d}{dt}\left(\frac{\partial v^2}{\partial \bar{l}}\right) = -\frac{a \sqrt{1 - e^2}\, \sin u}{(1 - e \cos u)^3} \frac{dL}{dt} \tag{42}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial v^1}{\partial \bar{a}}\right) = \frac{1}{(1 - e \cos u)} \frac{d^2 L}{dt^2} \left[-\sin u + \frac{3}{2} \frac{n(t - t_o)\,(\cos u - e)}{(1 - e \cos u)^2}\right]$$

$$+ \frac{3n(\cos u - e)}{(1 - e \cos u)^3} \frac{dL}{dt} - \frac{1}{(1 - e \cos u)^3} \left(\frac{dL}{dt}\right)^2$$

$$\times \left\{(\cos u - e) + \frac{3}{2} \frac{n(t - t_o)\,\sin u}{(1 - e \cos u)} \left[1 + \frac{3e(\cos u - e)}{(1 - e \cos u)}\right]\right\}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial v^2}{\partial \bar{a}}\right) = \frac{\sqrt{1 - e^2}}{(1 - e \cos u)} \frac{d^2 L}{dt^2} \left[\cos u + \frac{3}{2} \frac{n(t - t_o)\,\sin u}{(1 - e \cos u)^2}\right] \tag{43}$$

$$+ \frac{3n \sqrt{1 - e^2}\, \sin u}{(1 - e \cos u)^3} \frac{dL}{dt} - \frac{\sqrt{1 - e^2}}{(1 - e \cos u)^3} \left(\frac{dL}{dt}\right)^2$$

$$\times \left\{\sin u + \frac{3}{2} \frac{n(t - t_o)}{(1 - e \cos u)} \left[\frac{3e \sin^2 u}{(1 - e \cos u)} - \cos u\right]\right\}$$

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$$\frac{d^2}{dt^2}\left(\frac{\partial v^1}{\partial \overline{e}}\right) = -\frac{a\,\sin u}{(1-e\cos u)^2}\frac{d^2L}{dt^2}\left[\cos u + \frac{(\cos u - e)}{(1-e\cos u)}\right]$$

$$+ \frac{a}{(1-e\cos u)^4}\left(\frac{dL}{dt}\right)^2\left[-(\cos u - e)^2 - (1-e^2) + \frac{3\sin^2 u(1-e^2)}{(1-e\cos u)}\right]$$

$$\frac{d^2}{dt^2}\left(\frac{\partial v^2}{\partial \overline{e}}\right) = \frac{a}{(1-e\cos u)}\frac{d^2L}{dt^2}\left\{-\frac{e\cos u}{\sqrt{1-e^2}} + \frac{\sqrt{1-e^2}}{(1-e\cos u)}\left[\cos^2 u - \frac{\sin^2 u}{(1-e\cos u)}\right]\right\}$$

$$+ \frac{a}{(1-e\cos u)^3}\left(\frac{dL}{dt}\right)^2\left\{\frac{e\sin u}{\sqrt{1-e^2}} - \frac{\sqrt{1-e^2}}{(1-e\cos u)}\right.$$

$$\left.\times\left[4\sin u\cos u - \frac{3e\sin^3 u}{(1-e\cos u)}\right]\right\} \qquad (44)$$

$$\frac{d^2}{dt^2}\left(\frac{\partial v^1}{\partial \overline{\ell}}\right) = -\frac{a(\cos u - e)}{(1-e\cos u)^3}\frac{d^2L}{dt^2} + \frac{a\,\sin u}{(1-e\cos u)^4}\left(\frac{dL}{dt}\right)^2\left[1 + \frac{3e(\cos u - e)}{(1-e\cos u)}\right]$$

$$\frac{d^2}{dt^2}\left(\frac{\partial v^2}{\partial \overline{\ell}}\right) = -\frac{a\sqrt{1-e^2}\,\sin u}{(1-e\cos u)^3}\frac{d^2L}{dt^2} + \frac{a\sqrt{1-e^2}}{(1-e\cos u)^4}\left(\frac{dL}{dt}\right)^2$$

$$\times\left[-\cos u + \frac{3e\sin^2 u}{(1-e\cos u)}\right] \qquad (45)$$

Let $\overline{\alpha}$ denote one of the parameters $\overline{a}$, $\overline{e}$, $\overline{\ell}$. By (22) we then have the partial derivatives of the position, velocity and acceleration in Brown's lunar orbit with respect to $\overline{a}$, $\overline{e}$, $\overline{\ell}$ given by

$$\frac{\partial y^j}{\partial \overline{\alpha}} = \sum_{k=1}^{2} C_k^j \frac{\partial v^1}{\partial \overline{\alpha}}$$

$$\frac{d}{dt}\left(\frac{\partial y^j}{\partial \overline{\alpha}}\right) = \sum_{k=1}^{2} C_k^j \frac{d}{dt}\left(\frac{\partial v^k}{\partial \overline{\alpha}}\right) + \sum_{k=1}^{2}\frac{dC_k^j}{dt}\frac{\partial v^k}{\partial \overline{\alpha}}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial y^j}{\partial \overline{\alpha}}\right) = \sum_{k=1}^{2} C_k^j \frac{d^2}{dt^2}\left(\frac{\partial v^k}{\partial \overline{\alpha}}\right) + 2\sum_{k=1}^{2}\frac{dC_k^j}{dt}\frac{d}{dt}\left(\frac{\partial v^k}{\partial \overline{\alpha}}\right)$$

$$+ \sum_{k=1}^{2}\frac{d^2 C_k^j}{dt^2}\frac{\partial v^k}{\partial \overline{\alpha}}$$

$$j = 1, 2, 3 \qquad . \qquad (46)$$

Next, we define

i_0 = inclination of Brown's mean lunar orbital plane at time t_0 on the mean ecliptic of time t_0

Ω_0 = ascending node of Brown's mean lunar orbital plane on the mean ecliptic of time t_0 measured from the mean equinox of time t_0 along the ecliptic

ω_0 = argument of perigee of Brown's mean lunar orbit at time t_0 measured along the orbital plane from the ascending node on the mean ecliptic of time t_0.

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Further, let $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$ be the similar reference angles at the initial time t_o of Brown's mean lunar orbit relative to the coordinate system (y^1, y^2, y^3) referred to the mean equinox and equator of 1950.0 as defined at the beginning of this section. Preliminary to finding the partial derivatives of the position, velocity and acceleration in Brown's mean lunar orbit with respect to $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$, we shall derive the expressions for the partial derivatives of i_o, Ω_o, ω_o with respect to $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$.

By (20) and the fact that the matrix A is orthogonal, we can write

$$B_k^j = \sum_{\ell=1}^{3} A_j^\ell C_k^\ell \quad , \quad j, k = 1, 2, 3 \tag{47}$$

where the matrix B is given by (11) with the angles i, Ω, ω replaced by i_o, Ω_o, ω_o, the matrix C is given by (11) with B_k^j replaced by C_k^j and with the angles i, Ω, ω replaced by $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$, and the matrix A is given in Appendix B evaluated at time t_o. From (47) it follows that

$$\cos i_o = \sum_{\ell=1}^{3} A_3^\ell C_3^\ell \quad , \quad 0° \leqslant i_o \leqslant 180° \tag{48}$$

$$\left. \begin{aligned} \sin i_o \sin \Omega_o &= \sum_{\ell=1}^{3} A_1^\ell C_3^\ell \\[2ex] -\sin i_o \cos \Omega_o &= \sum_{\ell=1}^{3} A_2^\ell C_3^\ell \end{aligned} \right\} \quad 0° \leqslant \Omega_o < 360° \tag{49}$$

$$\left. \begin{aligned} \sin i_o \sin \omega_o &= \sum_{\ell=1}^{3} A_3^\ell C_1^\ell \\[2ex] \sin i_o \cos \omega_o &= \sum_{\ell=1}^{3} A_3^\ell C_2^\ell \end{aligned} \right\} \quad 0° \leqslant \omega_o < 360° \quad . \tag{50}$$

Let $\bar{\alpha}$ denote one of the parameters $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$. Differentiating (48) we obtain

$$\frac{\partial i_o}{\partial \bar{\alpha}} = -\frac{1}{\sin i_o} \sum_{\ell=1}^{3} A_3^\ell \frac{\partial C_3^\ell}{\partial \bar{\alpha}} \quad . \tag{51}$$

Differentiating the equations in (49), we obtain

$$\sin i_o \cos \Omega_o \frac{\partial \Omega_o}{\partial \bar{\alpha}} + \cos i_o \sin \Omega_o \frac{\partial i_o}{\partial \bar{\alpha}} = \sum_{\ell=1}^{3} A_1^\ell \frac{\partial C_3^\ell}{\partial \bar{\alpha}}$$

$$\sin i_o \sin \Omega_o \frac{\partial \Omega_o}{\partial \bar{\alpha}} - \cos i_o \cos \Omega_o \frac{\partial i_o}{\partial \bar{\alpha}} = \sum_{\ell=1}^{3} A_2^\ell \frac{\partial C_3^\ell}{\partial \bar{\alpha}} \quad .$$

Multiplying the first of these equations by $\cos \Omega_o$ and the second by $\sin \Omega_o$ and adding, we see that

$$\frac{\partial \Omega_o}{\partial \bar{\alpha}} = \frac{1}{\sin i_o} \left(\cos \Omega_o \sum_{\ell=1}^{3} A_1^\ell \frac{\partial C_3^\ell}{\partial \bar{\alpha}} + \sin \Omega_o \sum_{\ell=1}^{3} A_2^\ell \frac{\partial C_3^\ell}{\partial \bar{\alpha}} \right) \quad . \tag{52}$$

Differentiating the equations in (50), we obtain

$$\sin i_o \cos \omega_o \frac{\partial \omega_o}{\partial \bar{\alpha}} + \cos i_o \sin \omega_o \frac{\partial i_o}{\partial \bar{\alpha}} = \sum_{\ell=1}^{3} A_3^\ell \frac{\partial C_1^\ell}{\partial \bar{\alpha}}$$

$$-\sin i_o \sin \omega_o \frac{\partial \omega_o}{\partial \bar{\alpha}} + \cos i_o \cos \omega_o \frac{\partial i_o}{\partial \bar{\alpha}} = \sum_{\ell=1}^{3} A_3^\ell \frac{\partial C_2^\ell}{\partial \bar{\alpha}} \quad .$$

Multiplying the first of these equations by $\cos \omega_o$ and the second by $\sin \omega_o$ and subtracting, we see that

$$\frac{\partial \omega_o}{\partial \bar{\alpha}} = \frac{1}{\sin i_o} \left(\cos \omega_o \sum_{\ell=1}^{3} A_3^\ell \frac{\partial C_1^\ell}{\partial \bar{\alpha}} - \sin \omega_o \sum_{\ell=1}^{3} A_3^\ell \frac{\partial C_2^\ell}{\partial \bar{\alpha}} \right) \quad . \tag{53}$$

Finally, using expression (11) for the matrix C with B_k^j replaced by C_k^j and the angles i, Ω, ω replaced by $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$, we see that

$$\frac{\partial C_k^1}{\partial \bar{\Omega}} = -C_k^2 \qquad \frac{\partial C_k^2}{\partial \bar{\Omega}} = C_k^1 \qquad \frac{\partial C_k^3}{\partial \bar{\Omega}} = 0 \quad , \quad k = 1, 2, 3 \tag{54}$$

$$\frac{\partial C_1^j}{\partial \bar{\omega}} = C_2^j \qquad \frac{\partial C_2^j}{\partial \bar{\omega}} = -C_1^j \qquad \frac{\partial C_3^j}{\partial \bar{\omega}} = 0 \quad , \quad j = 1, 2, 3 \tag{55}$$

$$\left. \begin{array}{ccc} \dfrac{\partial C_1^1}{\partial \bar{i}} = C_1^3 \sin \bar{\Omega} & \dfrac{\partial C_2^1}{\partial \bar{i}} = C_2^3 \sin \bar{\Omega} & \dfrac{\partial C_3^1}{\partial \bar{i}} = \sin \bar{\Omega} \cos \bar{i} \\[2mm] \dfrac{\partial C_1^2}{\partial \bar{i}} = -C_1^3 \cos \bar{\Omega} & \dfrac{\partial C_2^2}{\partial \bar{i}} = -C_2^3 \cos \bar{\Omega} & \dfrac{\partial C_3^2}{\partial \bar{i}} = -\cos \bar{\Omega} \cos \bar{i} \\[2mm] \dfrac{\partial C_1^3}{\partial \bar{i}} = \sin \bar{\omega} \cos \bar{i} & \dfrac{\partial C_2^3}{\partial \bar{i}} = \cos \bar{\omega} \cos \bar{i} & \dfrac{\partial C_3^3}{\partial \bar{i}} = -\sin \bar{i} \end{array} \right\} \quad . \tag{56}$$

In (56) the angles $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$ are determined by

$$\cos \bar{i} = C_3^3 \quad , \quad 0° \leqslant \bar{i} \leqslant 180° \tag{57}$$

$$\left. \begin{array}{l} \sin \bar{i} \sin \bar{\Omega} = C_3^1 \\[3mm] -\sin \bar{i} \cos \bar{\Omega} = C_3^2 \end{array} \right\} \quad 0° \leqslant \bar{\Omega} < 360° \tag{58}$$

$$\left. \begin{array}{l} \sin \bar{i} \sin \bar{\omega} = C_1^3 \\[3mm] \sin \bar{i} \cos \bar{\omega} = C_2^3 \end{array} \right\} \quad 0° \leqslant \bar{\omega} < 360° \quad . \tag{59}$$

Let $(\alpha_o^1, \alpha_o^2, \alpha_o^3)$ denote the quantities $(i_o, \Omega_o, \omega_o)$ and let $(\overline{\alpha}^1, \overline{\alpha}^2, \overline{\alpha}^3)$ denote the quantities $(\overline{i}, \overline{\Omega}, \overline{\omega})$. Then by (22) the partial derivatives of the position, velocity and acceleration in Brown's mean lunar orbit with respect to the initial angles $\overline{i}$, $\overline{\Omega}$, $\overline{\omega}$ are

$$
\left.
\begin{aligned}
\frac{\partial y^j}{\partial \overline{\alpha}^\ell} &= \sum_{m=1}^{3} \frac{\partial \alpha_o^m}{\partial \overline{\alpha}^\ell} \sum_{k=1}^{2} \frac{\partial C_k^{j\cdot}}{\partial \alpha_o^{in}} v^k \\[2mm]
\frac{d}{dt}\left(\frac{\partial y^j}{\partial \overline{\alpha}^\ell}\right) &= \sum_{m=1}^{3} \frac{\partial \alpha_o^m}{\partial \overline{\alpha}^\ell} \left[\sum_{k=1}^{3} \frac{\partial C_k^j}{\partial \alpha_o^m} \frac{dv^k}{dt} + \sum_{k=1}^{2} \frac{d}{dt}\left(\frac{\partial C_k^j}{\partial \alpha_o^m}\right) v^k \right] \\[2mm]
\frac{d^2}{dt^2}\left(\frac{\partial y^j}{\partial \overline{\alpha}^\ell}\right) &= \sum_{m=1}^{3} \frac{\partial \alpha_o^m}{\partial \overline{\alpha}^\ell} \left[\sum_{k=1}^{2} \frac{\partial C_k^j}{\partial \alpha_o^m} \frac{d^2 v^k}{dt^2} + 2 \sum_{k=1}^{2} \frac{d}{dt}\left(\frac{\partial C_k^j}{\partial \alpha_o^m}\right) \frac{dv^k}{dt} \right. \\[2mm]
&\qquad \left. + \sum_{k=1}^{2} \frac{d^2}{dt^2}\left(\frac{\partial C_k^j}{\partial \alpha_o^m}\right) v^k \right].
\end{aligned}
\right\}
\quad
\begin{aligned}
j &= 1, 2, 3 \\
\ell &= 1, 2, 3
\end{aligned}
\qquad (60)
$$

where the 3×3 matrix $(\partial \alpha_o^m / \partial \overline{\alpha}^\ell)$ is determined from formulas (51), (52) and (53) evaluated at the initial time t_o. By (20) and (21) we have

$$
\left.
\begin{aligned}
\frac{\partial C_k^j}{\partial \alpha_o^m} &= \sum_{\ell=1}^{3} A_\ell^j \frac{\partial B_k^\ell}{\partial \alpha_o^m} \\[2mm]
\frac{d}{dt}\left(\frac{\partial C_k^j}{\partial \alpha_o^m}\right) &= \sum_{\ell=1}^{3} \frac{dA_\ell^j}{dt} \frac{\partial B_k^\ell}{\partial \alpha_o^m} + \sum_{\ell=1}^{3} A_\ell^j \frac{\partial}{\partial \alpha_o^m}\left(\frac{dB_k^\ell}{dt}\right) \\[2mm]
\frac{d^2}{dt^2}\left(\frac{\partial C_k^j}{\partial \alpha_o^m}\right) &= \sum_{\ell=1}^{3} \frac{d^2 A_\ell^j}{dt^2} \frac{\partial B_k^\ell}{\partial \alpha_o^m} + 2 \sum_{\ell=1}^{3} \frac{dA_\ell^j}{dt} \frac{\partial}{\partial \alpha_o^m}\left(\frac{dB_k^\ell}{dt}\right) \\[2mm]
&\qquad + \sum_{\ell=1}^{3} A_\ell^j \frac{\partial}{\partial \alpha_o^m}\left(\frac{d^2 B_k^\ell}{dt^2}\right)
\end{aligned}
\right\}
\quad
\begin{aligned}
j &= 1, 2, 3 \\
k &= 1, 2 \\
m &= 1, 2, 3
\end{aligned} \quad .
\qquad (61)
$$

Let α^1, α^2, α^3 denote i, Ω, ω. Then by (6) we can write

$$
\alpha^j = \alpha_o^j + \alpha_1^j(t - t_o) + \alpha_2^j(t - t_o)^2 + \alpha_3^j(t - t_o)^3
$$

so that

$$
\frac{\partial B_k^\ell}{\partial \alpha_o^j} = \frac{\partial B_k^\ell}{\partial \alpha^j} \quad , \qquad \ell = 1, 2, 3 \quad , \qquad k = 1, 2 \quad .
\qquad (62)
$$

In (62) the partial derivatives with respect to $\alpha^2 = \Omega$ and $\alpha^3 = \omega$ are given by (13) and (14), and the partial derivatives with respect to $\alpha^3 = i$ are given by

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$$\frac{\partial B_1^1}{\partial i} = B_1^3 \sin \Omega \quad \frac{\partial B_1^2}{\partial i} = -B_1^3 \cos \Omega \quad \frac{\partial B_1^3}{\partial i} = \sin \omega \cos i \left.\vphantom{\frac{\partial B_1^1}{\partial i}}\right\}$$

$$\frac{\partial B_2^1}{\partial i} = B_2^3 \sin \Omega \quad \frac{\partial B_2^2}{\partial i} = -B_2^3 \cos \Omega \quad \frac{\partial B_2^3}{\partial i} = \cos \omega \cos i \left.\vphantom{\frac{\partial B_2^1}{\partial i}}\right\} \qquad (63)$$

By (12) and (62), we have

$$\left.\begin{aligned}
\frac{\partial}{\partial \alpha_o^j}\left(\frac{dB_k^\ell}{dt}\right) &= \frac{\partial}{\partial \Omega}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\frac{d\Omega}{dt} + \frac{\partial}{\partial \omega}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\frac{d\omega}{dt} \\[2mm]
\frac{\partial}{\partial \alpha_o^j}\left(\frac{d^2 B_k^\ell}{dt^2}\right) &= \frac{\partial}{\partial \Omega}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\frac{d^2\Omega}{dt^2} + \frac{\partial}{\partial \omega}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\frac{d^2\omega}{dt^2} \\[2mm]
&\quad + \frac{\partial^2}{\partial \Omega^2}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\left(\frac{d\Omega}{dt}\right)^2 + \frac{\partial^2}{\partial \omega^2}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\left(\frac{d\omega}{dt}\right)^2 \\[2mm]
&\quad + 2\,\frac{\partial^2}{\partial \Omega\,\partial \omega}\left(\frac{\partial B_k^\ell}{\partial \alpha^j}\right)\frac{d\Omega}{dt}\,\frac{d\omega}{dt}
\end{aligned}\right\}
\begin{aligned}
& j = 1, 2, 3 \\
& k = 1, 2 \\
& \ell = 1, 2, 3
\end{aligned} \qquad (64)$$

where the partial derivatives of $\partial B_k^\ell / \partial \alpha^j$ are given by (13) through (17) with B_k^ℓ replaced by $\partial B_k^\ell / \partial \alpha^j$.

V. DETERMINATION OF MOTION AND PARTIAL DERIVATIVES WITH RESPECT TO INITIAL CONDITIONS

Let $(x_{me}^1, \ldots, x_{me}^6)$ denote the position and velocity of the Moon relative to the Earth, and let $(y_{me}^1, \ldots, y_{me}^9)$ denote the position, velocity and acceleration in Brown's mean lunar orbit. Let

$$\xi_{me}^k = x_{me}^k - y_{me}^k \quad , \quad k = 1, \ldots, 6 \quad . \qquad (65)$$

Then by (4) the $(\xi_{me}^1, \ldots, \xi_{me}^6)$ satisfy the system of equations

$$\left.\begin{aligned}
\frac{d\xi_{me}^k}{dt} &= \xi_{me}^{k+3} \\[2mm]
\frac{d\xi_{me}^{k+3}}{dt} &= -\gamma M_s\left(\frac{M_c}{M_s}\right)\frac{x_{me}^k}{r_{me}^3} - y_{me}^{k+6} + B^k + \Psi^k \\[2mm]
&\quad + H^k + \left(\frac{1}{M_m}F_m^k - \frac{1}{M_e}F_e^k\right) \\[2mm]
\xi_{me}^k &= \xi_{ome}^k \quad , \quad \xi_{me}^{k+3} = \xi_{ome}^{k+3} \quad \text{when } t = t_o
\end{aligned}\right\}
\quad k = 1, 2, 3 \qquad (66)$$

where

$$\xi_{ome}^k = (x_{me}^k - y_{me}^k)\big|_{t=t_o} \quad , \quad k = 1, \ldots, 6 \quad . \qquad (67)$$

PEP determines the position and velocity of the Moon as functions of time by numerically integrating the differential equation system (66) for the $(\xi^1_{me}, \ldots, \xi^6_{me})$ using relation (65) and the fact that the $(y^1_{me}, \ldots, y^9_{me})$ are known as functions of time. During the integration the positions of perturbing planets $1, 2, 4, \ldots, 9$ in the Ψ^k term of (3) are determined from an input magnetic tape. The position of the Earth-Moon barycenter relative to the Sun $(x^1_{cs}, x^2_{cs}, x^3_{cs})$, which is needed in evaluating the B^k term of (3) because of the relations[11]

$$\left.\begin{array}{l} x^k_{es} = x^k_{cs} - \dfrac{M_m}{M_c}\, x^k_{me} \\[3mm] x^k_{ms} = x^k_{cs} + \dfrac{M_m}{M_c}\, x^k_{me} \end{array}\right\} \quad k = 1, 2, 3 \tag{68}$$

is determined either from the perturbing planet input tape or from integrating the equations of motion of the Earth-Moon barycenter given in Ref. 12 along with equations (66).

Let $\partial x^k_{me}/\partial \beta^j_m$ $(j, k = 1, \ldots, 6)$ denote the partial derivatives of the position and velocity of the Moon with respect to initial osculating elliptic orbital elements $(\beta^1_m, \ldots, \beta^6_m)$, and let $\partial y^k_{me}/\partial \bar{\beta}^j_m$ $(j = 1, \ldots, 6,\ k = 1, \ldots, 9)$ denote the partial derivatives of the position, velocity and acceleration in Brown's mean lunar orbit with respect to initial mean orbital elements $(\bar{\beta}^1_m, \ldots, \bar{\beta}^6_m)$. Let

$$\eta^k_{mj} = \frac{\partial x^k_{me}}{\partial \beta^j_m} - \frac{\partial y^k_{me}}{\partial \bar{\beta}^j_m} \quad , \quad j, k = 1, \ldots, 6 \quad . \tag{69}$$

Then by (5) the η^k_{mj} $(j, k = 1, \ldots, 6)$ satisfy the system of equations

$$\left.\begin{array}{l} \dfrac{d\eta^k_{mj}}{dt} = \eta^{k+3}_{mj} \\[4mm] \dfrac{d\eta^{k+3}_{mj}}{dt} = \gamma M_s \left(\dfrac{M_c}{M_s}\right) \dfrac{1}{r^3_{me}} \left(\dfrac{3x^k_{me}}{r^2_{me}} \displaystyle\sum_{\ell=1}^{3} x^\ell_{me}\, \dfrac{\partial x^\ell_{me}}{\partial \beta^j_m} - \dfrac{\partial x^k_{me}}{\partial \beta^j_m}\right) \\[5mm] \qquad - \dfrac{\partial y^{k+6}_{me}}{\partial \bar{\beta}^j_m} + \dfrac{\partial B^k}{\partial \beta^j_m} + \dfrac{\partial \Psi^k}{\partial \beta^j_m} + \dfrac{\partial H^k}{\partial \beta^j_m} + \dfrac{\partial}{\partial \beta^j_m}\left(\dfrac{1}{M_m}\, F^k_m - \dfrac{1}{M_e}\, F^k_e\right) \\[5mm] \eta^k_{mj} = \eta^k_{omj} \quad , \quad \eta^{k+3}_{mj} = \eta^{k+3}_{omj} \quad \text{when } t = t_o \end{array}\right\} \begin{array}{l} k = 1, 2, 3 \\ j = 1, \ldots, 6 \end{array} \tag{70}$$

where

$$\eta^k_{omj} = \left(\frac{\partial x^k_{me}}{\partial \beta^j_m} - \frac{\partial y^k_{me}}{\partial \bar{\beta}^j_m}\right)\Bigg|_{t=t_o} \quad , \quad j, k = 1, \ldots, 6 \quad . \tag{71}$$

The $\partial x^k_{me}/\partial \beta^j_m$ at the initial time t_o are determined from the elliptic orbit formulas of Ref. 13.

PEP determines the $\partial x^k_{me}/\partial \beta^j_m$ $(j, k = 1, \ldots, 6)$ as functions of time either by assuming that they are equal to the $\partial y^k_{me}/\partial \bar{\beta}^j_m$ $(j, k = 1, \ldots, 6)$ or by numerically integrating the differential

equation system (70) for the η^k_{mj} (j, k = 1, ..., 6) using relation (69) and the fact that the $\partial y^k_{me}/$ $\partial \bar{\beta}^j_m$ (j = 1, ..., 6, k = 1, ..., 9) are known as functions of time. PEP has these two options, because it might be sufficiently accurate to assume that $\partial x^k_{me}/\partial \beta^j_m = \partial y^k_{me}/\partial \bar{\beta}^j_m$ in the least-squares process of fitting to observations. As mentioned earlier, not having to follow the exact procedure of numerically integrating equations (70) would save a great deal of computer time.

We intend to adjust the initial osculating elliptic orbital elements rather than the initial position and velocity in the least-squares process of fitting ephemerides to radar and optical observations, because the method of integration we use makes it easy to calculate the partial derivatives of position and velocity with respect to the former parameters and because the former parameters should be less correlated than the latter. The elliptic orbital elements $(\beta^1, ..., \beta^6)$ we have used in this report and in Ref. 1 are:

a = semimajor axis

e = eccentricity $(0 \leqslant e < 1)$

i = inclination $(0° \leqslant i \leqslant 180°)$

Ω = longitude of ascending node $(0° \leqslant \Omega < 360°)$

ω = argument of perigee, measured along the orbital plane from the ascending node $(0° \leqslant \omega < 360°)$

ℓ_o = initial mean anomaly .

Some other choice of elliptic orbital elements could be less correlated relative to the radar and optical data than this choice. For instance, this would very likely be the case if we replaced the angles (Ω, ω, ℓ_o) by the angles

$$
\left.
\begin{aligned}
\tilde{\Omega} &= \Omega \\
\tilde{\omega} &= \Omega + \omega \\
M &= \Omega + \omega + \ell_o
\end{aligned}
\right\}
\tag{72}
$$

since the angles in (72) are those generally used in celestial mechanics and were probably arrived at from the results of experience with optical data. The formulas in this report and in Ref. 1 are stated in terms of the angles (Ω, ω, ℓ_o). The partial derivatives with respect to $(\tilde{\Omega}, \tilde{\omega}, M)$ can be determined from these results because of the relations

$$
\left.
\begin{aligned}
\frac{\partial}{\partial \tilde{\Omega}} &= \frac{\partial}{\partial \Omega} - \frac{\partial}{\partial \omega} \\
\frac{\partial}{\partial \tilde{\omega}} &= \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \ell_o} \\
\frac{\partial}{\partial M} &= \frac{\partial}{\partial \ell_o}
\end{aligned}
\right\}
\cdot
\tag{73}
$$

EXPANSION FOR PRECESSION MATRIX

Let (y^1, y^2, y^3) be a coordinate system referred to the mean equinox and equator of 1950.0 (J. E. D. 2433282.423), and let (x^1, x^2, x^3) be a coordinate system referred to the mean equinox and equator of date. Then the relation between them is given by

$$\left. \begin{array}{c} y^j = \sum_{k=1}^{3} P_k^j x^k \\[2em] x^j = \sum_{k=1}^{3} P_j^k y^k \end{array} \right\} \quad j = 1, 2, 3 \qquad (A-1)$$

where the orthogonal matrix (P_k^j) is the precession matrix.

To give the established expression for the precession matrix, we first define the angles[14]

$$\left. \begin{array}{l} \zeta_0 = 2304\rlap{.}''948T + 0\rlap{.}''302T^2 + 0\rlap{.}''0179T^3 \\[1.5em] z = 2304\rlap{.}''948T + 1\rlap{.}''093T^2 + 0\rlap{.}''0192T^3 \\[1.5em] \Theta = 2004\rlap{.}''255T - 0\rlap{.}''426T^2 - 0\rlap{.}''0416T^3 \end{array} \right\} \qquad (A-2)$$

where T is measured in tropical centuries of 36524.21988 ephemeris days from the epoch 1950.0 (J. E. D. 2433282.423) to the instant of interest. Then the precession matrix at this instant is[15]

$$\left. \begin{array}{l} P_1^1 = \cos\zeta_0 \cos\Theta \cos z - \sin\zeta_0 \sin z \\[1em] P_1^2 = -\sin\zeta_0 \cos\Theta \cos z - \cos\zeta_0 \sin z \\[1em] P_1^3 = -\sin\Theta \cos z \\[1em] P_2^1 = \cos\zeta_0 \cos\Theta \sin z + \sin\zeta_0 \cos z \\[1em] P_2^2 = -\sin\zeta_0 \cos\Theta \sin z + \cos\zeta_0 \cos z \\[1em] P_2^3 = -\sin\Theta \sin z \\[1em] P_3^1 = \cos\zeta_0 \sin\Theta \\[1em] P_3^2 = -\sin\zeta_0 \sin\Theta \\[1em] P_3^3 = \cos\Theta \end{array} \right\} \qquad (A-3)$$

Let τ denote the time from the epoch 1950.0 (J. E. D. 2433282.423) in units of 10,000 ephemeris days. Then by Taylor's theorem we have

$$P_k^j = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \left. \frac{d^n P_k^j}{d\tau^n} \right|_{\tau=0} \right) \tau^n \quad , \quad j, k = 1, 2, 3 \quad . \tag{A-4}$$

Treating the coefficients in (A-2) as exact, some simple calculations show that the terms up to the fifth power in the Taylor expansions (A-4) are:

$$P_1^1 = 1.0 - 2.22603398052517 \times 10^{-5} \tau^2 - 2.6903385325366 \times 10^{-9} \tau^3$$
$$+ 8.191221606878 \times 10^{-11} \tau^4 + 1.79948222850 \times 10^{-14} \tau^5$$

$$P_1^2 = -6.119064710033514 \times 10^{-3} \tau - 5.06975739290688 \times 10^{-7} \tau^2$$
$$+ 4.5321716219079 \times 10^{-8} \tau^3 + 8.619581795926 \times 10^{-12} \tau^4$$
$$- 1.02943658327 \times 10^{-13} \tau^5$$

$$P_1^3 = -2.660399722772102 \times 10^{-3} \tau + 1.54818397804898 \times 10^{-7} \tau^2$$
$$+ 1.9729201591810 \times 10^{-8} \tau^3 + 1.960730253191 \times 10^{-12} \tau^4$$
$$- 4.39298354075 \times 10^{-14} \tau^5$$

$$P_2^1 = 6.119064710033514 \times 10^{-3} \tau + 5.06975739290688 \times 10^{-7} \tau^2$$
$$- 4.5321716219079 \times 10^{-8} \tau^3 - 9.636891635856 \times 10^{-12} \tau^4$$
$$+ 1.02604298897 \times 10^{-13} \tau^5$$

$$P_2^2 = 1.0 - 1.87214764627888 \times 10^{-5} \tau^2 - 3.1022173551368 \times 10^{-9} \tau^3$$
$$+ 6.882478825535 \times 10^{-11} \tau^4 + 1.91215207447 \times 10^{-14} \tau^5$$

$$P_2^3 = -8.13957902909886 \times 10^{-6} \tau^2 - 5.8309700675934 \times 10^{-10} \tau^3$$
$$+ 2.994360606802 \times 10^{-11} \tau^4 + 5.71739459043 \times 10^{-15} \tau^5$$

$$P_3^1 = 2.660399722772102 \times 10^{-3} \tau - 1.54818397804898 \times 10^{-7} \tau^2$$
$$- 1.9729201591810 \times 10^{-8} \tau^3 + 3.791379581151 \times 10^{-13} \tau^4$$
$$+ 4.50404085077 \times 10^{-14} \tau^5$$

$$P_3^2 = -8.13957902909886 \times 10^{-6} \tau^2 + 1.8168268497009 \times 10^{-10} \tau^3$$
$$+ 3.024323052660 \times 10^{-11} \tau^4 + 2.58550054981 \times 10^{-17} \tau^5$$

$$\cdot P_3^3 = 1.0 - 3.53886334246294 \times 10^{-6} \tau^2 + 4.1187882260017 \times 10^{-10} \tau^3$$
$$+ 1.308742781343 \times 10^{-11} \tau^4 - 1.12669845971 \times 10^{-15} \tau^5$$

$$\tag{A-5}$$

APPENDIX B
EXPANSION FOR MATRIX A AND ITS DERIVATIVES

The mean obliquity of the ecliptic is[16]

$$\epsilon_o = 23°27'08''26 - 46''845T - 0''0059T^2 + 0''00181T^3 \tag{B-1}$$

where T is measured in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 E. T. = J. E. D. 2415020.0 to the instant of interest. The relation between a coordinate system (x^1, x^2, x^3) referred to the mean equinox and equator of date and a coordinate system (w^1, w^2, w^3) referred to the mean equinox and ecliptic of date with the same origin is[17]

$$\left. \begin{array}{l} x^1 = w^1 \\[8pt] x^2 = w^2 \cos \epsilon_o - w^3 \sin \epsilon_o \\[8pt] x^3 = w^2 \sin \epsilon_o + w^3 \cos \epsilon_o \end{array} \right\} \quad . \tag{B-2}$$

Let τ denote the time from the epoch 1950.0 (J. E. D. 2433282.423) in units of 10,000 ephemeris days. By treating the coefficients in (B-1) as exact, Taylor expansions similar to (A-4) give

$$\left. \begin{array}{l} \sin \epsilon_o = 0.3978811865927521 - 5.70513893192403 \times 10^{-4}\tau \\[8pt] \qquad - 1.8312087506169 \times 10^{-9}\tau^2 + 1.652267540061 \times 10^{-10}\tau^3 \\[8pt] \qquad + 4.45783951328 \times 10^{-15}\tau^4 - 2.36469209 \times 10^{-19}\tau^5 \\[8pt] \cos \epsilon_o = 0.9174369522509674 + 2.47424898500217 \times 10^{-5}\tau \\[8pt] \qquad - 1.31335717\,30992 \times 10^{-9}\tau^2 - 7.173527734648 \times 10^{-11}\tau^3 \\[8pt] \qquad + 1.02732897621 \times 10^{-14}\tau^4 + 3.29806267 \times 10^{-19}\tau^5 \end{array} \right\} \quad . \tag{B-3}$$

Combining transformations (A-1) and (B-2) gives transformation (18), where

$$\left. \begin{array}{l} A_1^j = P_1^j \\[8pt] A_2^j = P_2^j \cos \epsilon_o + P_3^j \sin \epsilon_o \\[8pt] A_3^j = P_3^j \cos \epsilon_o - P_2^j \sin \epsilon_o \end{array} \right\} \quad j = 1, 2, 3 \quad . \tag{B-4}$$

The Taylor expansions for the A_1^j ($j = 1, 2, 3$) are given by the first three equations in (A-5). To determine the Taylor expansions for the A_2^j, A_3^j ($j = 1, 2, 3$), we multiply and add the expansions given in (A-5) and (B-3) as indicated in (B-4):

21

$$A_2^1 = 6.672379076707188 \times 10^{-3}\tau + 4.03140345445988 \times 10^{-7}\tau^2$$

$$- 4.9421227156709 \times 10^{-8}\tau^3 - 8.685948302421 \times 10^{-12}\tau^4$$

$$+ 1.11902061670 \times 10^{-13}\tau^5$$

$$A_2^2 = 0.9174369522509674 + 2.47424898500217 \times 10^{-5}\tau$$

$$- 2.04156730272966 \times 10^{-5}\tau^2 - 2.8443776398581 \times 10^{-9}\tau^3$$

$$+ 7.513826113715 \times 10^{-11}\tau^4 + 1.75327407517 \times 10^{-14}\tau^5$$

$$A_2^3 = 0.3978811865927521 - 5.70513893192403 \times 10^{-5}\tau$$

$$- 8.87742893170170 \times 10^{-6}\tau^2 - 2.0534553328574 \times 10^{-10}\tau^3$$

$$+ 3.266231487901 \times 10^{-11}\tau^4 + 4.79023554313 \times 10^{-15}\tau^5$$

$$A_3^1 = 6.08828576338671 \times 10^{-6}\tau + 7.11738284222153 \times 10^{-8}\tau^2$$

$$- 3.4836043645777 \times 10^{-11}\tau^3 - 9.238939614350 \times 10^{-14}\tau^4$$

$$- 1.72691125954 \times 10^{-16}\tau^5$$

$$A_3^2 = -0.3978811865927521 + 5.70513893192403 \times 10^{-5}\tau$$

$$- 1.67960985290526 \times 10^{-8}\tau^2 - 3.3710116720406 \times 10^{-11}\tau^3$$

$$+ 1.616276978960 \times 10^{-13}\tau^4 + 7.61970380671 \times 10^{-16}\tau^5$$

$$A_3^3 = 0.9174369522509674 + 2.47424898500217 \times 10^{-5}\tau$$

$$- 9.41199405263501 \times 10^{-9}\tau^2 - 1.3793679110716 \times 10^{-11}\tau^3$$

$$+ 6.983259897028 \times 10^{-14}\tau^4 + 3.21079987643 \times 10^{-16}\tau^5$$

$$(B-5)$$

The expansion for the matrix A given in (B-5) and the first three equations of (A-5) has 13 decimal place accuracy 30 years away from the epoch 1950.0 and 9 decimal place accuracy 300 years away from the epoch 1950.0. Actually, we treat expansions (A-5) and (B-5) as defining the matrix A so that the formulas for the mean lunar orbit involving the matrix A can be regarded as exact, except for those which assume that the inverse of A is equal to its transpose. But this is only done in computing the matrix $(\partial\alpha_o^m/\partial\bar{\alpha}^\ell)$ in (60), so that only those formulas involving partial derivatives with respect to initial mean orbital elements $(\bar{i}, \bar{\Omega}, \bar{\omega})$ are affected. Because of the use which is to be made of these quantities, any possible loss in accuracy due to the assumption that the inverse of A is equal to its transpose is unimportant.

Let t denote time measured in days. For a function f defined by

$$f = \sum_{n=0}^{5} f_n \tau^n$$

we have

$$\frac{df}{dt} = 10^{-4} \sum_{n=1}^{5} n f_n \tau^{n-1}$$

$$\frac{d^2 f}{dt^2} = 10^{-8} \sum_{n=2}^{5} n(n-1) f_n \tau^{n-2} \quad .$$

Thus, the coefficients in the expansions for dA/dt and d^2A/dt^2 are easily derived from the expansion for the matrix A given in (B-5) and the first three equations of (A-5).
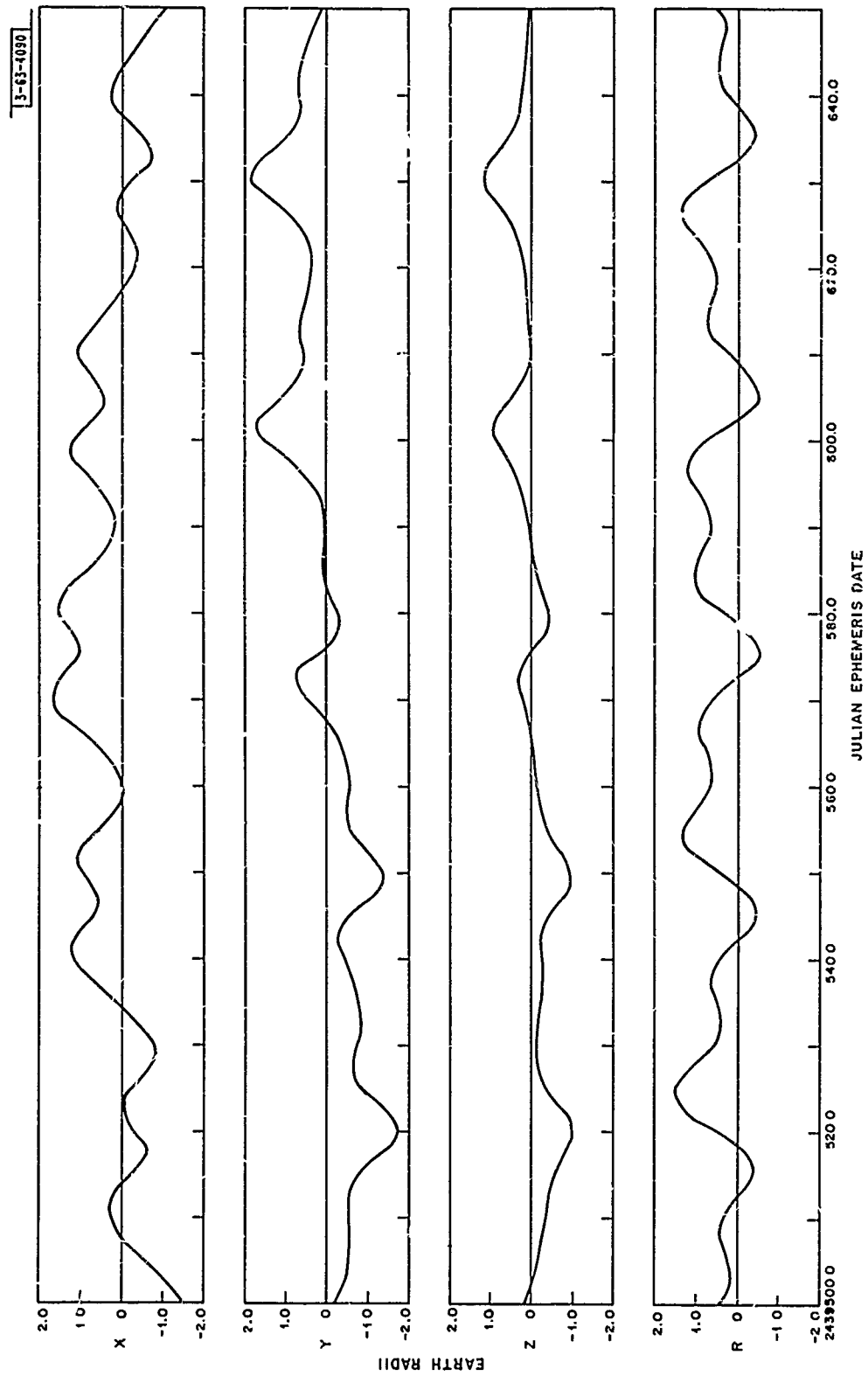
APPENDIX C
DIFFERENCE BETWEEN TRUE LUNAR ORBIT AND MEAN LUNAR ORBIT

The graphs of Fig. C-1(a-c) represent the differences between the x, y and z components and the radius vectors in the true lunar orbit and in Brown's mean lunar orbit during the 450-day period from 9 January 1967 to 3 April 1968. The distance unit is earth radii and the coordinate system is referred to the mean equinox and equator of 1950.0. The true lunar orbit coordinates were taken from the Jet Propulsion Laboratory Ephemeris Tapes;[18] the mean lunar orbit coordinates were evaluated by using the formulas in this report.

From these graphs it is clear that the mean lunar orbit does indeed follow the true orbit on the average as the Moon moves at a distance of 60 earth radii from the Earth. However, there are considerable oscillations about the mean orbit (mostly due to the Sun), so that numerically integrating the equations for the difference between the true and mean lunar orbits represents only a $1\frac{1}{2}$ order-of-magnitude improvement over numerically integrating the original equations of motion. Although this is a considerable saving, it is not as large as one might hope.

Brown's lunar theory represents the motion of the Moon by the mean lunar orbit plus over 1650 trigonometric terms. Adding a few of the larger trigonometric terms to the mean lunar orbit would yield an orbit which gives a further saving in representing the true orbit of the Moon. (From the graphs it would appear that the main terms of this addition would have periods of about 14 days and somewhat less than a year.) At some future time we shall alter the mean lunar orbit subroutines in PEP in this manner; for the present they utilize the formulas in this report.
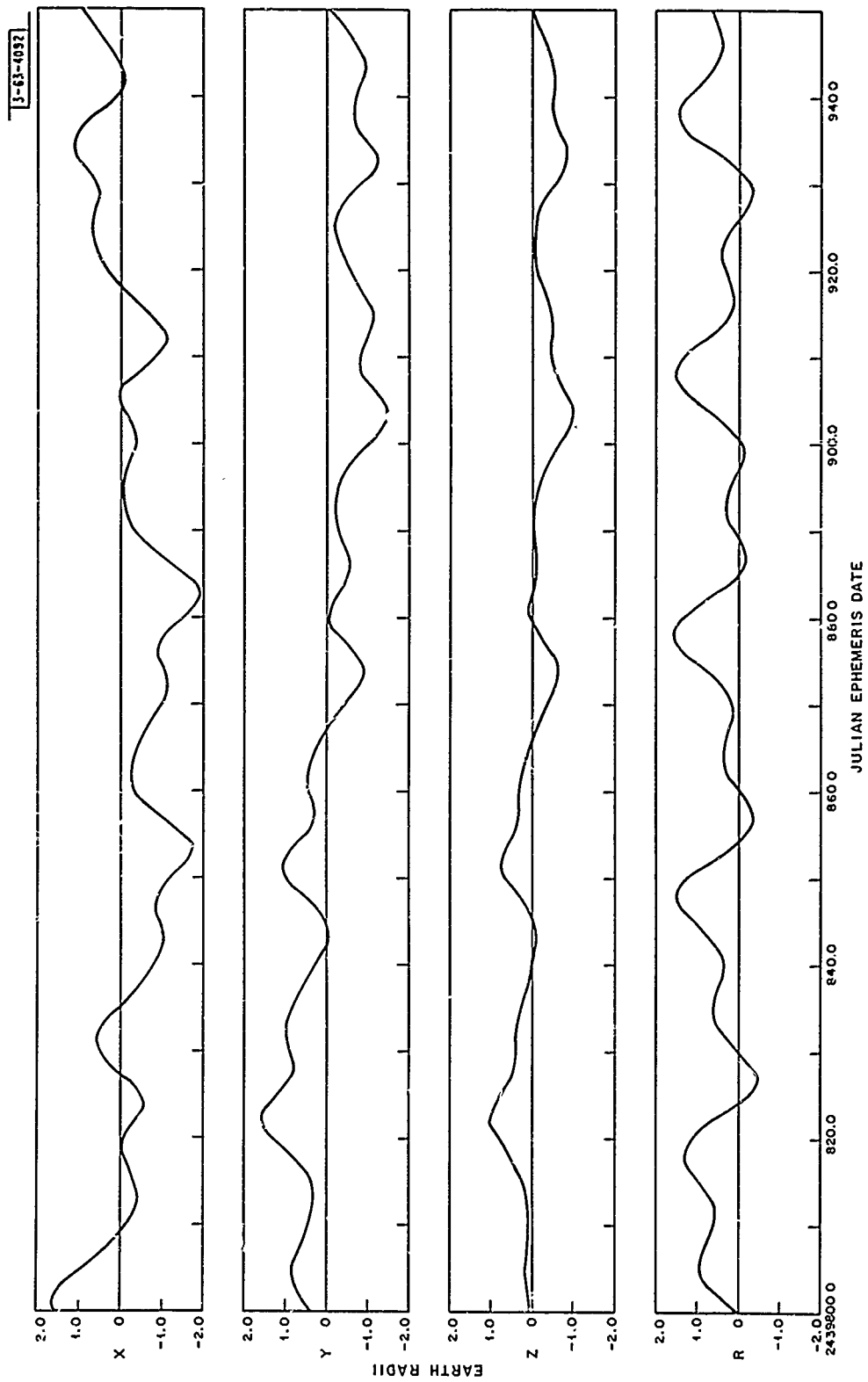
Fig. C-1(a-c). Difference between true lunar orbit and mean lunar orbit.
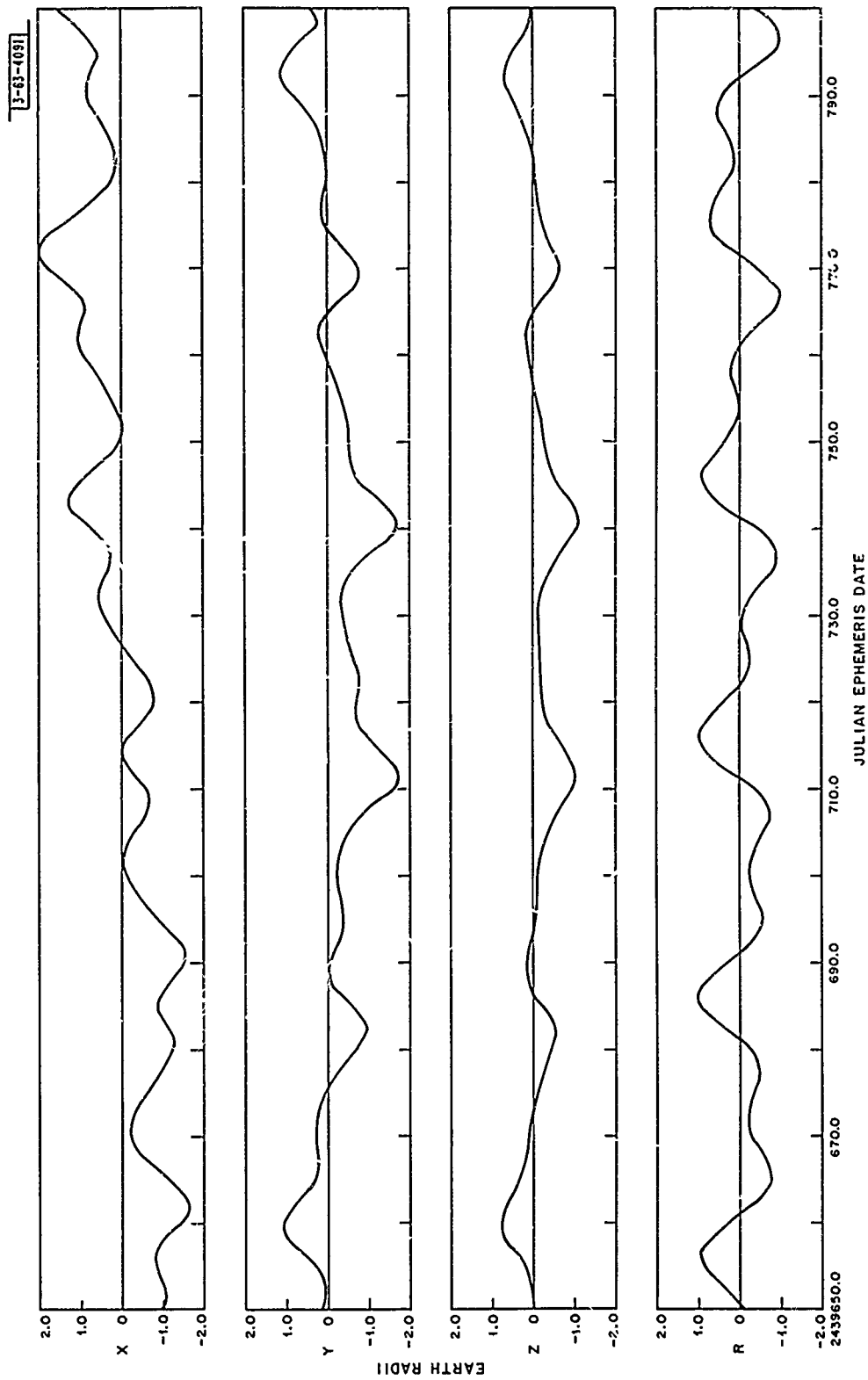
(a)

Fig. C-1. Continued.

(b)

JULIAN EPHEMERIS DATE

EARTH RADII

27

Fig. C-1. Continued.

(c)

28

ACKNOWLEDGMENT

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6. Ibid., Eq. (V-21).

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13. ABSTRACT

This report describes the method of integration that is used in a Lincoln Laboratory computer program (the Planetary Ephemeris Program) to determine as functions of time the position and velocity of the Moon and the partial derivatives of these quantities with respect to initial conditions. The method consists of numerically integrating the differential equations for the differences between the positions, velocities and partial derivatives in the true lunar orbit and in Brown's mean lunar orbit.

14. KEY WORDS

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Moon	precession